

MINIMAL FREE RESOLUTIONS OF MONOMIAL IDEALS ARE SUPPORTED ON POSETS

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ABSTRACT. We introduce the notion of a *resolution supported on a poset*. When the poset is a CW-poset, i.e. the face poset of a regular CW-complex, we recover the notion of cellular resolution. Work of Velasco has shown that there are monomial ideals whose minimal free resolutions are not cellular, hence cannot be supported on any CW-poset. We show that for any monomial ideal there is instead a *homology CW-poset* that supports the minimal free resolution of the ideal. In general there is more than one choice for the isomorphism class of such a poset, and it is an open question whether there is a canonical one.

INTRODUCTION

Understanding the structure of the maps in the minimal free resolution of a monomial ideal is a main open problem in commutative algebra. A common approach has been to use related combinatorial or topological objects such as simplicial complexes [2], CW-complexes [1, 3], or posets [9], to construct a template chain complex that would produce the desired minimal free resolution after an appropriate homogenization. However, Velasco [17] has shown that the chain complexes arising from simplicial complexes, and more generally CW-complexes, are not sufficiently flexible to accommodate all monomial ideals.

In this paper, we show that chain complexes arising from posets have the necessary degree of generality. We introduce the notion of a *resolution supported on a poset*, which is a refined and better behaved version of the notion of poset resolution from [9]. When the poset is a CW-poset, i.e. the face poset of a regular CW-complex, we recover the notion of cellular resolution [2]. In particular, Velasco's examples [17] imply that there are monomial ideals whose minimal free resolution cannot be supported on a CW-poset. This situation changes when one considers a more general class of posets. A finite poset P is a *homology CW-poset* (or *hCW-poset* for short) over a commutative ring with unit \mathbb{k} if for each $a \in P$ the filter $P_{<a} = \{x \in P \mid x < a\}$ is a homology sphere over \mathbb{k} , i.e. the reduced homology with coefficients in \mathbb{k} of the order simplicial complex $\Delta_{<a} = \Delta(P_{<a})$ is the same as that of a $(\dim \Delta_{<a})$ -sphere. In our main result, Theorem 6.1, we prove that every monomial ideal in a polynomial ring over a field \mathbb{k} has a minimal free resolution supported on an appropriate hCW-poset over \mathbb{k} . In general, such an hCW-poset is not unique, and it is an open question whether one can be constructed in a canonical way. Our proof uses a series of non-canonical choices, and we identify in Theorem 6.2 the class of ideals for which no choices arise. This turns out to be the class of *rigid* monomial ideals [10]. In this case the Betti poset [10, 11, 16] is an hCW-poset that supports the minimal free resolution, and this property characterizes rigid ideals.

Throughout this paper \mathbb{k} is a commutative ring with unit, rings are commutative \mathbb{k} -algebras with unit, modules are unitary, and unadorned tensor products are over \mathbb{k} .

1. THE CONIC CHAIN COMPLEX

Let P be a finite poset. For each $a \in P$ we consider the *half-open* filter $P_{\leq a} = \{x \in P \mid x \leq a\}$ and the *open* filter $P_{< a} = \{x \in P \mid x < a\}$; and their corresponding order complexes $\Delta_{\leq a} = \Delta(P_{\leq a})$ and $\Delta_{< a} = \Delta(P_{< a})$. We call the number $d_P(a) = d(a) = \dim(\Delta_{\leq a})$ the *dimension* of the element a of P . For each n we set $P^n = \{a \in P \mid d(a) \leq n\}$, we filter the order complex $\Delta = \Delta(P)$ of P by the subcomplexes $\Delta^n = \Delta(P^n) = \bigcup_{d(a) \leq n} \Delta_{\leq a}$, and we refer to Δ^n as the *conic n -skeleton* of $\Delta(P)$. We will always orient the faces of Δ by ordering their vertices in decreasing order (as defined in P). This induces a canonical filtration on the simplicial chain complex $C_\bullet(\Delta(P), \mathbb{k})$ over \mathbb{k} of the order complex $\Delta(P)$, and in turn a canonical fourth quadrant spectral sequence which has in first page as horizontal strands the complexes

$$\mathcal{C}_{q,\bullet}(P, \mathbb{k}) = 0 \longleftarrow H_0(\Delta^q, \Delta^{q-1}, \mathbb{k}) \longleftarrow \dots \longleftarrow H_n(\Delta^{n+q}, \Delta^{n+q-1}, \mathbb{k}) \longleftarrow \dots$$

at the $y = -q$ level. Of particular importance for us will be the strand at $q = 0$.

Definition 1.1. We set $\mathcal{C}_\bullet(P, \mathbb{k}) = \mathcal{C}_{0,\bullet}(P, \mathbb{k})$, and we call $\mathcal{C}_\bullet(P, \mathbb{k})$ the *conic chain complex* of P over \mathbb{k} .

Thus the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$ has the form

$$0 \leftarrow H_0(\Delta^0, \Delta^{-1}, \mathbb{k}) \leftarrow \dots \leftarrow H_{n-1}(\Delta^{n-1}, \Delta^{n-2}, \mathbb{k}) \xleftarrow{\partial_n} H_n(\Delta^n, \Delta^{n-1}, \mathbb{k}) \leftarrow \dots$$

and for $n \geq 1$ the map ∂_n is the composition $\partial_n = \iota_{n-1} \delta_n$ where ι_n and δ_n are the canonical maps $0 \rightarrow \tilde{H}_n(\Delta^n, \mathbb{k}) \xrightarrow{\iota_n} H_n(\Delta^n, \Delta^{n-1}, \mathbb{k}) \xrightarrow{\delta_n} \tilde{H}_{n-1}(\Delta^{n-1}, \mathbb{k}) \rightarrow \dots$ in the long exact sequence for relative homology of the pair (Δ^n, Δ^{n-1}) . In particular, since ι_n is injective for each n , we use it to identify $\tilde{H}_n(\Delta^n, \mathbb{k})$ as a \mathbb{k} -submodule of $H_n(\Delta^n, \Delta^{n-1}, \mathbb{k})$, and thus we obtain for $n \geq 1$ the important equality

$$(1.2) \quad \text{Ker } \partial_n = \tilde{H}_n(\Delta^n, \mathbb{k}).$$

Note that for any s , whenever $a \neq b$ with $d(a) = d(b) = s$, we must have an inclusion $\Delta_{\leq a} \cap \Delta_{\leq b} \subseteq \Delta^{s-1}$, and therefore

$$(1.3) \quad H_n(\Delta^{n+q}, \Delta^{n+q-1}, \mathbb{k}) = \bigoplus_{d(a)=n+q} H_n(\Delta_{\leq a}, \Delta_{< a}, \mathbb{k}) \cong \bigoplus_{d(a)=n+q} \tilde{H}_{n-1}(\Delta_{< a}, \mathbb{k})$$

Since the spectral sequence converges to the homology of $\Delta(P)$, we have the following basic result.

Proposition 1.4. *Suppose that for all $a \in P$ we have $\tilde{H}_m(\Delta(P_{< a}), \mathbb{k}) = 0$ for $m \leq d(a) - 2$. Then the simplicial chain complex $C_\bullet(\Delta(P), \mathbb{k})$ has the same homology as the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$.*

Proof. The assumptions on P in the proposition are equivalent to the condition $H_n(\Delta_{\leq a}, \Delta_{< a}, \mathbb{k}) = 0$ for $n \neq d(a)$. Thus, in view of (1.3), the spectral sequence collapses on the first page to the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$. \square

Next, we describe the relationship between $\mathcal{C}_\bullet(P, \mathbb{k})$ and the simplicial chain complex $C_\bullet(\Delta(P), \mathbb{k})$ in greater detail. Notice that, because of dimensional considerations, the submodule $Z_n(\Delta^n, \Delta^{n-1}, \mathbb{k})$ of relative n -cycles inside $C_n(\Delta^n, \mathbb{k})$ is isomorphic to $H_n(\Delta^n, \Delta^{n-1}, \mathbb{k})$. Combined with the canonical inclusion of oriented simplicial n -chains $C_n(\Delta^n, \mathbb{k}) \rightarrow C_n(\Delta, \mathbb{k})$, this induces for each n a canonical inclusion map of \mathbb{k} -modules $H_n(\Delta^n, \Delta^{n-1}, \mathbb{k}) \rightarrow C_n(\Delta, \mathbb{k})$ and it is straightforward to verify from the definitions that this identifies canonically the chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$ as a subcomplex of $C_\bullet(\Delta, \mathbb{k})$.

We will need an even more explicit description of the elements of $\mathcal{C}_\bullet(P, \mathbb{k})$ inside $C_\bullet(\Delta, \mathbb{k})$, and for this we introduce some notation. Recall that we orient the faces of $\Delta(P)$ by ordering their vertices in decreasing order. We write $[\]$ for the oriented empty face of Δ . If $\sigma = [a_0, \dots, a_n]$ is an oriented face of $\Delta_{<a}$, then we write $[a, \sigma]$ for the oriented face $[a, a_0, \dots, a_n]$ of $\Delta_{\leq a}$; in particular we have $[a, [\]] = [a]$. Also, if $w = \sum c_\sigma \sigma$ is an n -chain of $\Delta_{<a}$ then we write $[a, w]$ for the $(n+1)$ -chain of $\Delta_{\leq a}$

$$(1.5) \quad [a, w] = \sum c_\sigma [a, \sigma].$$

With this notation it is straightforward to verify that the elements of $\mathcal{C}_\bullet(P, \mathbb{k})$ belonging to a component $H_n(\Delta_{\leq a}, \Delta_{<a}, \mathbb{k})$ with $d(a) = n$ are exactly all elements of $C_n(\Delta, \mathbb{k})$ of the form $[a, z]$ where z is an $(n-1)$ -cycle of $\Delta_{<a}$. In particular, we have $\mathcal{C}_0(P, \mathbb{k}) = \bigoplus_{a \in P^0} \mathbb{k} \cdot [a]$, and therefore the boundary operator $C_0(\Delta, \mathbb{k}) \rightarrow C_{-1}(\Delta, \mathbb{k}) = \mathbb{k} \cdot [\]$ induces by restriction an augmentation map

$$\partial_0 : \mathcal{C}_0(P, \mathbb{k}) \rightarrow \mathbb{k} \cdot [\].$$

Definition 1.6. We set $\mathcal{C}_{-1}(P, \mathbb{k}) = \mathbb{k} \cdot [\]$ and we write $\tilde{\mathcal{C}}_\bullet(P, \mathbb{k})$ for the chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$ augmented with the map $\partial_0 : \mathcal{C}_0(P, \mathbb{k}) \rightarrow \mathcal{C}_{-1}(P, \mathbb{k})$.

Remarks 1.7. (a) It is now straightforward to check that (1.2) holds also for $n = 0$.

(b) It follows from the definitions that the chain complex $\bigoplus_{q \geq 0} \mathcal{C}_{q,\bullet}(P, \mathbb{k})$ is canonically isomorphic to the part in non-negative homological degrees of the shifted poset construction $\mathcal{D}_\bullet(\tilde{P}, \mathbb{k})(1)$ from [9], where \tilde{P} is the ranked poset obtained from P by using the canonical procedure from [9, Proposition A.9]. Thus the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$ should be thought of as a refined and better behaved version of $\mathcal{D}_\bullet(P, \mathbb{k})$. When the poset P satisfies the assumptions of Proposition 1.4 then we have a canonical isomorphism $\tilde{\mathcal{C}}_\bullet(P, \mathbb{k}) \cong \mathcal{D}_\bullet(\tilde{P}, \mathbb{k})(1)$. In particular, this holds when P is a homology CW-poset, hence in that case the conic chain complex is “essentially the same” as $\mathcal{D}_\bullet(P, \mathbb{k})$.

2. HOMOGENIZATION, AND RESOLUTIONS SUPPORTED ON POSETS

Let P be a poset structure on a set B . Consider a chain complex of \mathbb{k} -modules

$$\mathcal{F}_\bullet = 0 \leftarrow F_0 \leftarrow \dots \leftarrow F_{n-1} \xleftarrow{f_n} F_n \leftarrow \dots$$

A P -grading on \mathcal{F}_\bullet is a direct sum decomposition of \mathbb{k} -modules $F_n = \bigoplus_{a \in B} F_{n,a}$ for each n , such that for all $a, b \in B$ and all $x \in F_{n,a}$ the component $f_n^b(x)$ of $f_n(x)$ inside $F_{n-1,b}$ is zero when $b \not\leq a$ in P .

Examples 2.1. (a) For any poset P the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$ is naturally P -graded with the component $F_{n,a}$ zero when $n \neq d(a)$ and with $F_{d(a),a} = H_{d(a)}(\Delta_{\leq a}, \Delta_{<a}, \mathbb{k})$.

(b) For any poset P the poset construction chain complex $\mathcal{D}_\bullet(\tilde{P}, \mathbb{k})$, see [9], is naturally P -graded with $F_{n,a} = H_n(\Delta_{\leq a}, \Delta_{< a}, \mathbb{k})$, and when P is a homology CW-poset the canonical isomorphism from Remark 1.7(b) respects the P -grading.

(c) The cellular chain complex $C_\bullet(X, \mathbb{k})$ of a regular CW-complex X with coefficients in \mathbb{k} : here $F_n = H_n(X^n, X^{n-1}, \mathbb{k})$, the poset $P = P(X)$ is the face poset of X , and for an (open) face $\sigma \in P$ the component $F_{n,\sigma}$ is zero if $\dim \sigma \neq n$ and $F_{\dim \sigma, \sigma} = H_{\dim \sigma}(\bar{\sigma}, \dot{\sigma}, \mathbb{k})$. In this case $P = \tilde{P}$, and by construction the canonical isomorphism $C_\bullet(X, \mathbb{k}) \rightarrow \mathcal{D}_\bullet(P, \mathbb{k})(1)$ from [12, Theorem 1.7] respects the P -grading.

(d) For any poset P the simplicial chain complex $C_\bullet(\Delta(P), \mathbb{k})$ is naturally P -graded, where $F_{n,a}$ is the subspace of $C_n(\Delta(P), \mathbb{k})$ with basis all oriented n -faces of $\Delta(P)$ of the form $[a, a_1, \dots, a_n]$. In particular, the canonical inclusion $C_\bullet(P, \mathbb{k}) \rightarrow C_\bullet(\Delta(P), \mathbb{k})$ respects the P -grading.

Turning to the homological algebra of modules, consider a morphism of posets

$$\deg: P \rightarrow \mathbb{Z}^m.$$

For an element $\alpha \in \mathbb{Z}^m$ we denote by $P_{\deg \leq \alpha}$ the set of elements $x \in B$ such that $\deg(x) \leq \alpha$. We review the formalism of homogenizing a P -graded chain complex \mathcal{F}_\bullet with respect to the map \deg . This is a standard technique for studying resolutions of monomial ideals and multigraded modules, see e.g. [2, 3, 1, 14] for cases when each component $F_{n,a}$ is free of rank ≤ 1 , and [6, 15, 9] for other cases.

Let $R = \mathbb{k}[x_1, \dots, x_m]$ be a polynomial ring over \mathbb{k} with the standard \mathbb{Z}^m -grading. If $M = \bigoplus_{\alpha \in \mathbb{Z}^m} M_\alpha$ is a \mathbb{Z}^m -graded R -module and $\gamma \in \mathbb{Z}^m$ then $M(\gamma)$ is the \mathbb{Z}^m -graded R -module with $M(\gamma)_\alpha = M_{\gamma+\alpha}$. The *homogenization* of \mathcal{F}_\bullet (with respect to the map \deg) is the chain complex $\hat{\mathcal{F}}_\bullet$ of \mathbb{Z}^m -graded R -modules

$$\hat{\mathcal{F}}_\bullet = 0 \leftarrow \hat{F}_0 \leftarrow \dots \leftarrow \hat{F}_{n-1} \xleftarrow{\hat{f}_n} \hat{F}_n \leftarrow \dots,$$

where for each n we have $\hat{F}_n = \bigoplus_{a \in B} F_{n,a} \otimes_{\mathbb{k}} R(-\deg a)$, and for $x \in F_{n,a}$ one has

$$\hat{f}_n(x \otimes 1) = \sum_{b \leq a} f_n^b(x) \otimes x^{\deg a - \deg b}.$$

In particular, for $\alpha \in \mathbb{Z}^m$ the homogeneous component of \hat{F}_n of degree α is the \mathbb{k} -submodule

$$(2.2) \quad \hat{F}_{n,\alpha} = \bigoplus_{a \in P_{\deg \leq \alpha}} F_{n,a} \otimes x^{\alpha - \deg a}.$$

We are ready to introduce the main new notion of this paper.

Definition 2.3. We say that a chain complex \mathcal{F}_\bullet of \mathbb{Z}^m -graded R -modules is *supported on a poset* P if there is a map of posets $\deg: P \rightarrow \mathbb{Z}^m$ such that \mathcal{F}_\bullet is isomorphic to the homogenization of the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$.

Remark 2.4. There is a subtle distinction between the notion of poset resolution discussed in [9] and that of our definition of a complex supported on a poset. There exist \mathbb{Z}^m -graded free resolutions \mathcal{F}_\bullet which are poset resolutions with respect to a map of posets $\deg: P \rightarrow \mathbb{Z}^m$ but are not supported on the poset P in our sense.

Remark 2.5. Let X be a regular CW-complex with face poset P . By [4] the poset P is an hcw-poset over \mathbb{k} , hence Remark 1.7(b) and Examples 2.1(b) and 2.1(c) yield that the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$ and the cellular chain complex

$C_\bullet(X, \mathbb{k})$ are isomorphic as P -graded chain complexes of \mathbb{k} -modules. Therefore a chain complex \mathcal{F}_\bullet is supported on the regular CW-complex X in the sense of [3] exactly if it is supported on the face poset P in our sense.

The following standard fact will be needed later.

Proposition 2.6. *Let \mathbb{k} be a field. The homogenization of $\mathcal{C}_\bullet(P, \mathbb{k})$ is a resolution of a monomial ideal if and only if $\hat{\mathcal{C}}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})$ is exact whenever $P_{\deg \leq \alpha}$ is nonempty.*

Proof. From degree considerations it is clear that the homogenization $\hat{\mathcal{F}}_\bullet$ of $\mathcal{C}_\bullet(P, \mathbb{k})$ is a resolution precisely when each graded strand

$$\hat{\mathcal{F}}_{\bullet, \alpha} = 0 \longleftarrow \hat{F}_{0, \alpha} \longleftarrow \dots \longleftarrow \hat{F}_{n-1, \alpha} \xleftarrow{f_n} \hat{F}_{n, \alpha} \longleftarrow \dots$$

is a resolution. However, by (2.2) and the definitions, $\hat{\mathcal{F}}_{\bullet, \alpha}$ can be canonically identified with $\mathcal{C}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})$. \square

3. BASES WITH MINIMAL SUPPORT

Let R be a ring, and let

$$\mathcal{F}_\bullet = 0 \longleftarrow F_0 \longleftarrow \dots \longleftarrow F_{n-1} \xleftarrow{f_n} F_n \longleftarrow \dots \longleftarrow F_l \longleftarrow 0$$

be a length l chain complex of free R -modules of finite rank. Let $f_0: F_0 \rightarrow H_0(\mathcal{F}_\bullet)$ be the canonical projection. For each $0 \leq n \leq l$ let B_n be a basis of F_n , and let $B = \coprod B_n$ be their disjoint union. For any $y = \sum_{c \in B_n} a_c c \in F_n$ the *support* of y (with respect to B or B_n) is the set

$$\text{supp } y = \text{supp}_B y = \text{supp}_{B_n}(y) = \{c \in B_n \mid a_c \neq 0\}.$$

Definition 3.1. (a) When y is in F_n we say that y is a *cycle with minimal support* relative to the basis B if y is in $\text{Ker } f_n$ and the support of y does not contain properly the support of any nonzero element of $\text{Ker } f_n$.

(b) If $y \in F_n$ with $n \geq 1$ the support of $f_n(y)$ in F_{n-1} is called the *boundary support* of y .

(c) We say that B is a *basis with minimal (boundary) support* for \mathcal{F}_\bullet if for each $n \geq 1$ and each $b \in B_n$ the element $f_n(b) \in F_{n-1}$ is a cycle with minimal support relative to the basis B .

Remarks 3.2. (a) It is clear that if we replace each $b \in B$ by an associate b' , then the resulting bijective correspondence between B and the new basis B' commutes with taking supports and is an isomorphism of incidence posets (see Definition 4.1). In particular B' is a basis with minimal support if and only if B is.

(b) It is a standard exercise in linear algebra that when R is a field and \mathcal{F}_\bullet is a resolution of $H_0(\mathcal{F}_\bullet)$ then \mathcal{F}_\bullet has a basis with minimal support.

(c) The *simple syzygies* of Charalambous and Thoma [7, 8] are a toric analogue for the bases with minimal support of monomial ideals.

The following three lemmas are presumably familiar to experts, and we include their routine proofs for completeness. Throughout all arguments, we assume $R = \mathbb{k}[x_1, \dots, x_m]$ is a polynomial ring over a field \mathbb{k} with the standard \mathbb{Z}^m -grading.

Lemma 3.3. *Suppose \mathcal{F}_\bullet is a \mathbb{Z}^m -graded minimal free resolution of a finitely generated \mathbb{Z}^m -graded R -module M . Let B be a basis of \mathcal{F}_\bullet consisting of homogeneous*

elements. Let $\overline{\mathcal{F}}_\bullet = \mathcal{F}_\bullet \otimes_R R/(x_1 - 1, \dots, x_m - 1)$, and let \overline{B} be the induced by B basis of $\overline{\mathcal{F}}_\bullet$ over \mathbb{k} . Then:

- (a) For each $n \geq 1$ and each $\bar{b} \in \overline{B}$ one has $\bar{f}_n(\bar{b}) \neq 0$.
- (b) If \overline{B} is a basis with minimal support then so is B .
- (c) If B is a basis with minimal support and M is torsion-free then \overline{B} is a basis with minimal support.

Proof. Part (a) is immediate from the minimality of the resolution \mathcal{F}_\bullet . To prove the remaining parts we will use the well known property that going modulo the ideal $(x_1 - 1, \dots, x_m - 1)$ is an exact operation on \mathbb{Z}^m -graded modules, in particular one has that $\overline{\text{Ker } f_n} = \text{Ker } \bar{f}_n$. Part (b) is now straightforward from the fact that for each homogeneous element $y \in F_n$ one has that $\overline{\text{supp}_B y} = \text{supp}_{\overline{B}} \bar{y}$. Part (c) follows from the fact that if $\bar{z} = \sum_{\bar{b} \in \text{supp}(\bar{z})} a_{\bar{b}} \bar{b}$ (where each $a_{\bar{b}} \in \mathbb{k}$) is in $\text{Ker } \bar{f}_n$ then $z = \sum_{\bar{b} \in \text{supp}(\bar{z})} a_{\bar{b}} x^{\alpha - \deg b} b$ is in $\text{Ker } f_n$ (when $n = 0$ we use the torsion-freeness of M for this), where $\alpha \in \mathbb{Z}^m$ is the coordinate-wise maximum of the degrees $\deg b \in \mathbb{Z}^m$ with $\bar{b} \in \text{supp } \bar{z}$. \square

Lemma 3.4. *Let B be a homogeneous basis of minimal support for a minimal \mathbb{Z}^m -graded resolution \mathcal{F}_\bullet of a finitely generated torsion-free \mathbb{Z}^m -graded R -module M . Suppose $y \in F_n$ with $n \geq 1$ is a homogeneous element such that $\text{supp } f_n(y) = \text{supp } f_n(b)$ for some $b \in B_n$. Then for some homogeneous nonzero $a \in R$ we have $a f_n(b) = f_n(y)$. In particular, every two elements in B_n have non-comparable boundary supports.*

Proof. Let $f_n(b) = \sum_{e \in \text{supp } f_n(b)} a_e x^{\deg b - \deg e} e$. We first point out that $\deg b$ is the coordinate-wise maximum α of all $\deg e \in \mathbb{Z}^m$ with $e \in \text{supp } f_n(b)$. Indeed, we always have $\deg b \geq \alpha$, and if the inequality were strict we would have $f_n(b) = x^\beta w$ for some $w \in F_{n-1}$, and $\beta > 0$. Since \mathcal{F}_\bullet is a resolution and M is torsion-free it follows that $w = f_n(u)$ for some $u \in F_n$, hence $b - x^\beta u$ is in $\text{Ker } f_n \subseteq \mathfrak{m} F_n$, where $\mathfrak{m} = (x_1, \dots, x_m)$ is the maximal ideal of R generated by all the variables. It follows that $b \in \mathfrak{m} F_n$ which contradicts the minimality of our resolution.

Next, we prove that $f_n(y) = a \sum_{e \in \text{supp } f_n(b)} a_e x^{\deg b - \deg e} e$ for some homogeneous $0 \neq a \in R$. Indeed, suppose $f_n(y) = \sum_{e \in \text{supp } f_n(b)} c_e x^{\deg y - \deg e} e$ and note that since $\text{supp } f_n(y) = \text{supp } f_n(b)$ we must have $c_e \neq 0$ for all e . Pick an $e \in \text{supp } f_n(b)$. Then $f_n(a_e y - c_e x^{\deg y - \deg b} b)$ has support strictly contained in $\text{supp}(f_n(b))$. By the minimality of the support of $f_n(b)$ it follows that $f_n(a_e y - c_e x^{\deg y - \deg b} b) = 0$, and the desired conclusion is immediate.

Finally, if two elements b_1 and b_2 of B_n have comparable boundary supports, then by the minimality of boundary supports these two supports must be equal. By the first part of this proof we then must have that both our basis elements have the same degree. Therefore their images in F_{n-1} must differ by a multiple of a nonzero element $a \in \mathbb{k}$. Therefore $a b_1 - b_2 \in \text{Ker } f_n \subseteq \mathfrak{m} F_n$, so $\{b_1, b_2\}$ is linearly dependent modulo \mathfrak{m} , contradicting the minimality of the resolution \mathcal{F}_\bullet . \square

Lemma 3.5. *Suppose \mathcal{F}_\bullet is a \mathbb{Z}^m -graded minimal free resolution of a finitely generated torsion-free \mathbb{Z}^m -graded R -module M . Let C be any basis of F_0 consisting of homogeneous elements.*

Then \mathcal{F}_\bullet has a basis with minimal support B consisting of homogeneous elements and such that $B_0 = C$.

Iterating this procedure finitely many times yields the desired homogeneous basis with minimal support B_{k+1} . \square

When $\text{char}(\mathbb{k}) = 2$, the ideal I_Δ has total Betti numbers $(10, 15, 7, 1)$. The choice of basis B for the minimal free resolution of I_Δ given by Macaulay2 [13] yields

$$\begin{array}{c}
\begin{array}{c} S^{10} \leftarrow \begin{bmatrix} x_4 & x_5 & 0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & x_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_4 & 0 & 0 & x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_3 & 0 & 0 & 0 & x_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & x_3 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & 0 & x_4 \end{bmatrix} \\
\hline
d_1
\end{array} \\
\begin{array}{c} S^{15} \leftarrow \begin{bmatrix} x_5 & x_6 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & x_6 & 0 & 0 & 0 & x_4 x_6 \\ x_3 & 0 & 0 & x_6 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & x_6 & 0 & x_2 x_6 \\ x_1 & 0 & 0 & 0 & 0 & x_6 & 0 \\ 0 & x_4 & x_5 & 0 & 0 & 0 & x_4 x_5 \\ 0 & x_3 & 0 & x_5 & 0 & 0 & x_3 x_5 \\ 0 & x_2 & 0 & 0 & x_5 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & x_5 & 0 \\ 0 & 0 & x_3 & x_4 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & x_4 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & x_2 & x_3 & 0 & x_2 x_3 \\ 0 & 0 & 0 & x_1 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & 0 \end{bmatrix} \\
\hline
d_2
\end{array} \\
\begin{array}{c} S^7 \leftarrow \begin{bmatrix} x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \\ 0 \end{bmatrix} \\
\hline
d_3
\end{array} S
\end{array}$$

Example 3.7. Although bases of minimal support always exist, they are not unique. Consider the ideal $M = (uvw, uxy, wvyz, tuvxz, twxyz)$ in the polynomial ring $R = \mathbb{k}[t, u, v, w, x, y, z]$. Due to degree constraints, all homogeneous basis elements for the free modules in a minimal \mathbb{Z}^7 -graded free resolution of M are uniquely determined up to associates, except the basis element corresponding to the direct summand $R(-\mathbf{1})$ that appears in homological degree two of the resolution (in general, we write $\mathbf{1} = (1^m) = (1, \dots, 1, \dots, 1) \in \mathbb{Z}^m$). With one choice of this basis element, corresponding to the rightmost column of the map ∂_2 displayed below, we have a basis element with minimal boundary support consisting of three

elements:

$$R^5 \xleftarrow[\partial_1]{\begin{bmatrix} -xy & -yz & 0 & -txz & 0 & 0 \\ vw & 0 & -wz & 0 & -tvz & 0 \\ 0 & v & x & 0 & 0 & -tvx \\ 0 & 0 & 0 & w & y & 0 \\ 0 & 0 & 0 & 0 & 0 & u \end{bmatrix}} R^6 \xleftarrow[\partial_2]{\begin{bmatrix} z & tz \\ -x & 0 \\ v & 0 \\ 0 & -y \\ 0 & w \\ 0 & 0 \end{bmatrix}} R^2 \leftarrow 0.$$

With the second choice shown below, we have a basis element with minimal boundary support consisting of four elements:

$$R^5 \xleftarrow[\partial_1]{\begin{bmatrix} -xy & -yz & 0 & -txz & 0 & 0 \\ vw & 0 & -wz & 0 & -tvz & 0 \\ 0 & v & x & 0 & 0 & -tvx \\ 0 & 0 & 0 & w & y & 0 \\ 0 & 0 & 0 & 0 & 0 & u \end{bmatrix}} R^6 \xleftarrow[\partial_2]{\begin{bmatrix} z & 0 \\ -x & tx \\ v & -tv \\ 0 & -y \\ 0 & w \\ 0 & 0 \end{bmatrix}} R^2 \leftarrow 0.$$

Thus there are at least two substantially different choices for a basis with minimal support for the minimal free resolution of M .

Remark 3.8. The case where there is (up to associates) only one homogeneous basis (and hence no choices arise) is naturally of interest. Recall that a monomial ideal is *rigid*, see e.g. [10, 11], if the following two conditions on its \mathbb{Z}^m -graded Betti numbers hold:

- (R1) $\beta_{i,\alpha}$ is either 1 or 0 for all i and all $\alpha \in \mathbb{Z}^m$; and
- (R2) If $\beta_{i,\alpha} = 1$ and $\beta_{i,\alpha'} = 1$ then α and α' are incomparable in \mathbb{Z}^m .

By [10, Proposition 1.5] the minimal resolution of a monomial ideal has a unique up to associates \mathbb{Z}^m -graded basis if and only if the ideal is rigid. In particular, the unique \mathbb{Z}^m -graded basis in the minimal resolution of a rigid ideal is necessarily a basis with minimal support.

4. INCIDENCE POSETS

Let the ring R , the chain complex \mathcal{F}_\bullet , and the basis B be as in the beginning of the previous section. For each $b \in B_n$ with $n \geq 1$ we can write

$$f_n(b) = \sum_{c \in B_{n-1}} [b : c]c,$$

and we call the uniquely determined coefficient $[b : c] \in R$ the *incidence coefficient* of b and c (with respect to the chosen bases B_n and B_{n-1}).

Definition 4.1. We introduce a poset structure $P(\mathcal{F}_\bullet, B)$ on the set B by taking the partial ordering generated by the relations $c < b \iff [b : c] \neq 0$. We call this poset the *incidence poset* of \mathcal{F}_\bullet (with respect to the chosen basis B). When R is a polynomial ring over a field, $I \subseteq R$ is a monomial ideal, \mathcal{F}_\bullet a minimal \mathbb{Z}^m -graded free resolution of I , and B a homogeneous basis, we write $P(I, B)$ for $P(\mathcal{F}_\bullet, B)$ and call it the *incidence poset of I with respect to B* .

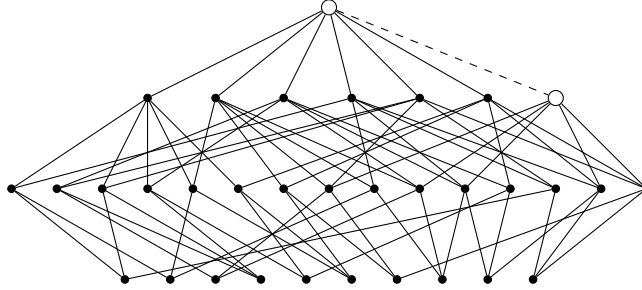
Remarks 4.2. Let $P = P(\mathcal{F}_\bullet, B)$.

- (a) \mathcal{F}_\bullet is naturally P -graded with $F_{n,a} = Ra$ for $a \in B_n$, and $F_{n,a} = 0$ otherwise.
- (b) Suppose $f_n(b) \neq 0$ for each $n \geq 1$ and each $b \in B_n$. It is a straightforward consequence of the definition that for every $n \geq 0$ and every $b \in B_n$ we have $d(b) = \dim \Delta(P_{\leq b}) = n$, and when $a \leq b$ every maximal chain in P that starts with a and ends with b has length $d(b) - d(a)$. In particular, the set B_0 is the set of minimal elements of P .

(c) Suppose that for some $n \geq 1$ we have $f_n \neq 0$ and the elements of B_n have non-comparable boundary supports. Then $f_n(b) \neq 0$ for each $b \in B_n$. Indeed, if $f_n(b) = 0$ for some $b \in B_n$ then the boundary support of b is the empty set, hence comparable to the boundary support of every other element of B_n , which forces $B_n = \{b\}$ and contradicts the assumption $f_n \neq 0$.

(d) Suppose that R is a domain, that for each $n \geq 1$ the elements of B_n have non-comparable boundary supports, B is a basis with minimal support, and let $b \in B_0$ be such that $f_0(b) = 0$. If $b' \in B_1$ is such that $b \in \text{supp } f_1(b')$ then we must have $\{b\} = \text{supp } f_1(b')$, otherwise $\text{supp } f_1(b') \setminus \{b\}$ would be the support of a nonzero element of $\text{Ker } f_0$ contradicting the minimal support property of B . Furthermore, there can be at most one such b' , otherwise we would have basis elements with comparable boundary supports. Finally, for such a b' we always must have $b' \notin \text{supp } f_2(b'')$ for any $b'' \in B_2$, otherwise a linear combination of elements from $B_1 \setminus \{b'\}$ would be mapped by f_1 to a nonzero multiple of b contradicting the fact that b is not in the boundary support of any of the elements from $B_1 \setminus \{b'\}$. In particular, P is the disjoint union of the posets $P' = P \setminus \{b, b'\}$ and $\{b, b'\}$, we have a direct sum decomposition $\mathcal{F}_\bullet = \mathcal{F}'_\bullet \oplus (0 \leftarrow Rb \leftarrow Rb' \leftarrow 0)$ where \mathcal{F}'_\bullet is the free subcomplex of \mathcal{F}_\bullet with basis $B' = B \setminus \{b, b'\}$, and B' is a basis with minimal support for \mathcal{F}'_\bullet such that $P' = P(\mathcal{F}'_\bullet, B')$ and such that the elements of B'_n with $n \geq 1$ have non-comparable boundary supports.

Example 4.3. The incidence poset $P(I_\Delta, B)$ for the homogeneous basis B of the minimal resolution of the Stanley-Reisner ideal I_Δ of the projective plane presented in Example 3.6 has Hasse diagram given by the solid edges below:



Note that the order complex of the open filter for the maximal element of dimension 3 is isomorphic to the first barycentric subdivision of Δ .

Example 4.4. In general, different choices of bases with minimal support for the same chain complex will result in non-isomorphic incidence posets. For instance, the two choices of a basis with minimal support from Example 3.7 give rise to two non-isomorphic incidence posets, with Hasse diagrams displayed below.



The following theorem provides a key technical tool needed for the proofs of our main results.

Theorem 4.5. *Let \mathbb{k} be a field, M a \mathbb{k} -module, and let B be a basis for a length l finite free resolution \mathcal{F}_\bullet of M over \mathbb{k} with $\dim_{\mathbb{k}} M = 1$. Let $P = P(\mathcal{F}_\bullet, B)$ and*

suppose for each $1 \leq n \leq l$ that $f_n \neq 0$ and that the elements of B_n have non-comparable boundary supports. The following are equivalent:

- (1) B is a basis with minimal support;
- (2) For each n and $a \in B_n$ we have $\dim_{\mathbb{k}} H_n(\Delta(P_{\leq a}), \Delta(P_{< a}), \mathbb{k}) = 1$, and there is an isomorphism of P -graded chain complexes

$$\phi_{\bullet} : \mathcal{F}_{\bullet} \longrightarrow \mathcal{C}_{\bullet}(P, \mathbb{k})$$

such that $\phi_n(a) \in H_n(\Delta(P_{\leq a}), \Delta(P_{< a}), \mathbb{k})$ for each $a \in B_n$.

Proof. Note that our assumptions imply, by Remarks 4.2(a) and (b), that the set of minimal elements of P is precisely B_0 , and, more generally, that $B_n = \{a \in B \mid d(a) = n\}$. Thus we have

$$\mathcal{C}_n(P, \mathbb{k}) = H_n(\Delta^n, \Delta^{n-1}, \mathbb{k}) = \bigoplus_{a \in B_n} H_n(\Delta(P_{\leq a}), \Delta(P_{< a}), \mathbb{k}).$$

Furthermore, by construction the differential ∂_n of the conic chain complex $\mathcal{C}_{\bullet}(P, \mathbb{k})$ maps $H_n(\Delta(P_{\leq a}), \Delta(P_{< a}), \mathbb{k})$ isomorphically onto $K_{n-1}^a = \tilde{H}_{n-1}(\Delta(P_{< a}), \mathbb{k})$ when $n \geq 1$; and this is exactly the kernel of ∂_{n-1} restricted to

$$\bigoplus_{a > b \in B_{n-1}} H_{n-1}(\Delta(P_{\leq b}), \Delta(P_{< b}), \mathbb{k}).$$

We now show that (2) implies (1). Indeed, suppose B is a basis which does not have minimal support. Then for some $n \geq 1$ and some $a \in B_n$ there is a non-zero cycle $c \in F_{n-1}$ with $\text{supp } c \subsetneq \text{supp } f_n(a)$. Therefore the kernel of f_{n-1} restricted to $\bigoplus_{a > b \in B_{n-1}} \mathbb{k}b$ is at least two-dimensional. Now the isomorphism ϕ_{\bullet} yields that K_{n-1}^a has to be also at least two-dimensional, a contradiction.

Next we proceed with the proof that (1) implies (2). We note that if \mathcal{F}_{\bullet} is of the form $0 \longleftarrow \mathbb{k}b \longleftarrow 0$ or $0 \longleftarrow \mathbb{k}b \longleftarrow \mathbb{k}b' \longleftarrow 0$ then our statement is trivially true. Since the conic chain complex construction turns a finite disjoint union of posets into a finite direct sum of conic chain complexes, in view of Remark 4.2(c) and Remark 3.2 we may assume without loss of generality also that $f_0(b)$ takes the same nonzero value for each $b \in B_0$. We will prove that

$$(4.6) \quad \dim_{\mathbb{k}} H_n(\Delta(P_{\leq a}), \Delta(P_{< a}), \mathbb{k}) = 1$$

for all $a \in B_n$ and construct the isomorphisms ϕ_n by using induction on n . Since P^{-1} is empty, we get that $\Delta^{-1}(P) = \Delta(P^{-1})$ is the empty simplicial complex and therefore $H_0(\Delta^0, \Delta^{-1}, \mathbb{k}) = H_0(\Delta^0, \mathbb{k}) = \bigoplus_{a \in B_0} \mathbb{k}[a]$. Since $F_0 = \bigoplus_{a \in B_0} \mathbb{k}a$, we define ϕ_0 by $\phi_0(a) = [a]$ and note that, since $f_0(b)$ takes the same nonzero value for each $b \in B_0$, the map ϕ_0 sends $\text{Ker } f_0$ isomorphically onto $\tilde{H}_0(\Delta^0, \mathbb{k})$.

Now assume $n \geq 1$ and that for all $0 \leq k \leq n-1$ we have already established (4.6) and have constructed isomorphisms ϕ_k as required in (2), and such that $\text{Ker } f_k$ gets mapped isomorphically onto $\tilde{H}_k(\Delta^k, \mathbb{k})$. Since $b \in B_{n-1}$ is smaller than $a \in B_n$ exactly when b is in the boundary support of a , the already constructed isomorphism ϕ_{n-1} shows that K_{n-1}^a is isomorphic to the kernel of f_{n-1} restricted to $\bigoplus_{b \in \text{supp } f_n(a)} \mathbb{k}b$. The minimal support condition on B now yields that K_{n-1}^a has dimension 1, and is spanned by $\phi_{n-1}(f_n(a))$. This completes the proof of (4.6), and we can now define ϕ_n by sending each $a \in B_n$ to the unique nonzero element in $H_n(\Delta(P_{\leq a}), \Delta(P_{< a}), \mathbb{k})$ that gets mapped under ∂_n to $\phi_{n-1}(f_n(a))$. The remaining desired properties of ϕ_n are now immediate. \square

Theorem 4.7. *Let \mathbb{k} be a field, let $R = \mathbb{k}[x_1, \dots, x_m]$ be a polynomial ring over \mathbb{k} , let I be a monomial ideal in R , and let \mathcal{F}_\bullet be a minimal \mathbb{Z}^m -graded free resolution of I over R . Suppose that B is a homogeneous basis of \mathcal{F}_\bullet with minimal support, and let $\deg : P(I, B) \rightarrow \mathbb{Z}^m$ be the map that assigns to each element of B its \mathbb{Z}^m -degree as an element of \mathcal{F}_\bullet .*

Then \deg is a morphism of posets, and the conic chain complex $\mathcal{C}_\bullet(P(I, B), \mathbb{k})$ produces a minimal free resolution of I after homogenization.

Proof. That \deg is a morphism of posets is immediate from the minimality of the resolution and the fact that the differential of \mathcal{F}_\bullet preserves the \mathbb{Z}^m -grading.

Let $P = P(I, B)$, let $\overline{\mathcal{F}}_\bullet = \mathcal{F}_\bullet / (x_1 - 1, \dots, x_m - 1)$, and let \overline{B} be the induced by B basis of $\overline{\mathcal{F}}_\bullet$. It is straightforward to observe that we have $P(I, B) = P(\overline{\mathcal{F}}_\bullet, \overline{B})$. Thus by Theorem 4.5 we have an isomorphism $\overline{\phi}_\bullet : \overline{\mathcal{F}}_\bullet \rightarrow \mathcal{C}_\bullet(P, \mathbb{k})$ of P -graded chain complexes which therefore lifts to an isomorphism of the corresponding homogenized chain complexes. Since the homogenization of $\overline{\mathcal{F}}_\bullet$ is \mathcal{F}_\bullet , we obtain the desired conclusion. \square

5. HOMOLOGY CW-POSETS

Recall that a finite poset P is called a *homology CW-poset* or *hCW-poset* over \mathbb{k} if for each $a \in P$ the open filter $P_{<a}$ is a homology sphere over \mathbb{k} .

Examples 5.1. (a) It is straightforward to check that the two incidence posets from Example 4.4 are hCW-posets.

(b) In general one cannot expect that incidence posets will be hCW-posets. For instance, the open filter of the dimension 3 element in the incidence poset from Example 4.3 is homeomorphic to the barycentric subdivision of the projective plane, hence has nonzero homology in two dimensions. This incidence poset is therefore not an hCW-poset.

Lemma 5.2. *Let $n \geq 0$, let $\deg : B \rightarrow \mathbb{Z}^m$ be a function, and let P be a poset structure on the finite set B such that \deg is a morphism of posets from P to \mathbb{Z}^m . Let $a \in B$ with $d_P(a) \geq n + 2$ and $\alpha = \deg(a)$, and suppose that for all $b \in P$ with $d_P(b) < d_P(a)$ the open filter $P_{<b}$ is a homology sphere over \mathbb{k} . Also, suppose that $H_n(\tilde{\mathcal{C}}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})) = 0$. Then there exists a poset structure P' on B such that the following conditions are satisfied:*

- (1) *The maps $\text{id}_B : P \rightarrow P'$ and $\deg : P' \rightarrow \mathbb{Z}^m$ are morphisms of posets.*
- (2) *For each $c \in B$ such that $a \not\leq_P c$ we have $P_{\leq c} = P'_{\leq c}$.*
- (3) *For each $c \in B$ we have $\dim \Delta(P_{\leq c}) = \dim \Delta(P'_{\leq c})$.*
- (4) *$\tilde{\mathcal{C}}_\bullet(P, \mathbb{k}) = \tilde{\mathcal{C}}_\bullet(P', \mathbb{k})$.*
- (5) *For $k \geq n + 1$ we have $\tilde{H}_k(\Delta(P_{<a}), \mathbb{k}) = \tilde{H}_k(\Delta(P'_{<a}), \mathbb{k})$.*
- (6) *$\tilde{H}_n(\Delta(P'_{<a}), \mathbb{k}) = 0$.*

In fact, the poset P' is obtained by creating extra edges extending down from the element a in the Hasse diagram of P .

Proof. We proceed by induction on $r = \dim_{\mathbb{k}} \tilde{H}_n(\Delta(P_{<a}), \mathbb{k})$. If $r = 0$ then we can simply take $P' = P$. Assume that $r \geq 1$ and let h be a non-zero element of $\tilde{H}_n(\Delta(P_{<a}), \mathbb{k})$. The first key observation is that when $m \leq n$ every element g of

$\tilde{H}_m(\Delta(P_{<a}), \mathbb{k})$ can be represented by an m -cycle z of the form

$$(5.3) \quad z = \sum_{c \in A_z} [c, z_c],$$

where A_z is an antichain in $P_{<a}$ with $d(c) = m$ for each $c \in A_z$, and each $(m-1)$ -chain z_c is a cycle of $\Delta(P_{<c})$. Indeed, any m -chain w of $\Delta(P_{<a})$ can be uniquely written in the form

$$(5.4) \quad w = \sum_{c \in A_w} [c, w_c],$$

for some subset A_w of $P_{<a}$ and $(m-1)$ -chains w_c of $\Delta(P_{<c})$. Let $k = \max\{d(c) \mid c \in A_w\}$, and let c_1, \dots, c_l be the elements of A_w of dimension k . If w is a cycle, this forces each w_{c_i} to be a cycle. If in addition $k > m$, then by our assumptions w_{c_i} is also a boundary of $\Delta(P_{<c_i})$, thus for each i we have a chain v_i of $\Delta(P_{<c_i})$ such that $\partial(v_i) = w_{c_i}$. Therefore, if $k > m$ and w represents g then the cycle $w' = w + \partial(\sum_{i=1}^l [c_i, v_i])$ also represents g and has $\max\{d(c) \mid c \in A_{w'}\} < k$. Iterating this procedure we arrive at the desired cycle z .

Second, we note that considered as a cycle of $\Delta(P)$, the cycle z of the form (5.3) that represents h is an element of $\tilde{\mathcal{C}}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})$. Since we also assume $H_n(\tilde{\mathcal{C}}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})) = 0$, this yields an $(n+1)$ -chain t in $\tilde{\mathcal{C}}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})$ such that $\partial(t) = z$, in particular $d(c) = n+1$ and t_c is a cycle for each $c \in A_t$.

Third, select the chain t above so that the non-empty set $C = A_t \setminus P_{<a}$ is minimal. Make a new poset structure P'' on the set B by adding to the relations from P the new relations $c < a$ for $c \in C$ and taking the poset they generate. It is now straightforward to check that conditions (1) through (4) of the Lemma are satisfied for P'' . Furthermore since in $P''_{<a}$ the elements of C are maximal but of dimension $n+1 < d(a)$, we get that $\tilde{H}_k(\Delta(P_{<a}), \mathbb{k}) = \tilde{H}_k(\Delta(P''_{<a}), \mathbb{k})$ for $k \geq n+2$.

Next, the minimality of C implies that an $(n+1)$ -cycle of z of $\Delta(P''_{<a})$ in the form (5.3) is in fact a cycle of $\Delta(P_{<a})$. Indeed, if there is $c \in A_z \cap C$ then z_c and t_c are both n -cycles in the n -dimensional homology sphere $\Delta(P''_{<c}) = \Delta(P_{<c})$ hence are the same up to a scalar multiple $s \in \mathbb{k}$. Therefore the chain $t - sz$ contradicts the minimality of the choice of t . It follows that $\tilde{H}_{n+1}(\Delta(P''_{<a}), \mathbb{k}) = \tilde{H}_{n+1}(\Delta(P_{<a}), \mathbb{k})$.

Finally, every element of $\tilde{H}_n(\Delta(P''_{<a}), \mathbb{k})$ can be represented by a cycle of the form (5.3) and all these are also cycles of $\Delta(P_{<a})$. Therefore the map $\text{id}_B : P \rightarrow P''$ induces a surjection $\tilde{H}_n(\Delta(P_{<a}), \mathbb{k}) \rightarrow \tilde{H}_n(\Delta(P''_{<a}), \mathbb{k})$ that has the non-zero element h in its kernel. Now we are done by applying our induction hypothesis to the poset P'' . \square

Applying this lemma repeatedly for all $a \in B$ in increasing order of $d(a)$ and $n = d(a) - 2, d(a) - 3, \dots, 0$, we obtain the following key consequence.

Theorem 5.5. *Let $\deg : B \rightarrow \mathbb{Z}^m$ be a function, and let P be a poset structure on the finite set B such that \deg is a morphism of posets from P to \mathbb{Z}^m , for each $a \in B$ one has $\dim_{\mathbb{k}} \tilde{H}_{d_P(a)-1}(\Delta(P_{<a}), \mathbb{k}) = 1$, and such that $\tilde{\mathcal{C}}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})$ is exact whenever $P_{\deg \leq \alpha}$ is non-empty. Then there exists an hcw-poset structure Q on the set B such that*

- (1) $\text{id}_B : P \rightarrow Q$ and $\deg : Q \rightarrow \mathbb{Z}^m$ are morphisms of posets;
- (2) for each $a \in B$ we have $\dim \Delta(P_{\leq a}) = \dim \Delta(Q_{\leq a})$; and
- (3) $\tilde{\mathcal{C}}_\bullet(P, \mathbb{k}) = \tilde{\mathcal{C}}_\bullet(Q, \mathbb{k})$. \square

Remark 5.6. If $P_1 = P(I, B')$ and $P_2 = P(I, B'')$ are two non-isomorphic incidence posets for the same monomial ideal I , then in general any hcw-posets Q_1 and Q_2 produced from them by applying the procedure from the proof of Theorem 5.5 will also be non-isomorphic. A trivial example for this is when the incidence posets are already hcw, like the two posets from Example 5.1(a).

Example 5.7. Applying Theorem 5.5 to the incidence poset $P(I_\Delta, B)$ from Example 5.1(b) produces an hcw-poset with Hasse diagram obtained by adding the single new dashed edge to the solid edges of the Hasse diagram displayed in Example 4.3.

6. MAIN RESULTS

We are now ready to prove the main result of this paper:

Theorem 6.1. *Let \mathbb{k} be a field, and let I be a monomial ideal in $R = \mathbb{k}[x_1, \dots, x_m]$. There exists an hcw-poset Q and a morphism of posets $\deg: Q \rightarrow \mathbb{Z}^m$ such that the minimal free \mathbb{Z}^m -graded resolution of I over R is supported on Q .*

Proof. Let \mathcal{F}_\bullet be a minimal free \mathbb{Z}^m -graded resolution of I over R . By Lemma 3.5 there exists a homogeneous basis B of minimal support for \mathcal{F}_\bullet . Let $\deg: B \rightarrow \mathbb{Z}^m$ be the map that assigns to each element of B its \mathbb{Z}^m -degree as an element of \mathcal{F}_\bullet . By Theorem 4.7, for the incidence poset $P = P(\mathcal{F}_\bullet, B)$ we have that $\deg: P \rightarrow \mathbb{Z}^m$ is a morphism of posets, and the conic chain complex $\mathcal{C}_\bullet(P, \mathbb{k})$ produces (up to isomorphism) after homogenization the resolution \mathcal{F}_\bullet . In particular, the chain complex $\tilde{\mathcal{C}}_\bullet(P_{\deg \leq \alpha}, \mathbb{k})$ is exact whenever $P_{\deg \leq \alpha}$ is nonempty. Since $d(a) = n$ for every $a \in B_n$ by Remark 4.2, it follows from Theorem 4.5 that $\dim_{\mathbb{k}} \tilde{H}_{d(a)-1}(\Delta(P_{<a}), \mathbb{k}) = 1$ for each $a \in P$. Therefore Theorem 5.5 yields the desired hcw-poset structure Q on the set B . \square

In general, the isomorphism class of the hcw-poset Q that supports a minimal free resolution of the monomial ideal I is not unique. When I belongs to the class of rigid ideals, as observed in Remark 3.8 there is a unique choice for a basis of minimal support, and the degree constraints on the basis elements yield that the corresponding incidence poset is exactly the Betti poset of I over \mathbb{k} . (The Betti poset of a monomial ideal I over \mathbb{k} is the set B_I of the \mathbb{Z}^m -degrees of the elements in a homogeneous basis of the minimal free \mathbb{Z}^m -graded resolution of I . See [10, 11, 16] for more on Betti posets.) It turns out that for rigid ideals the Betti poset is already an hcw-poset, and that this property characterizes rigid ideals:

Theorem 6.2. *The Betti poset of a monomial ideal is an hcw-poset if and only if the ideal is rigid.*

Proof. Let $P = B_I$ be the Betti poset of a monomial ideal I , and suppose first P is an hcw-poset. Rigidity condition (R1) is automatically satisfied, since $\beta_{a, d(a)-1} = 1$ holds for all $a \in P$. If condition (R2) did not hold, then we could find two comparable elements $a < b$ in \mathbb{Z}^m where $\beta_{a,i} = \beta_{b,i} = 1$ for some i . Under this assumption, the comparability $a < b$ would also hold in P . However, $a < b$ implies that $d(a) < d(b)$ and $\tilde{H}_{d(a)-1}(\Delta(P_{<a}), \mathbb{k}) \cong \mathbb{k}$ and $\tilde{H}_{d(b)-1}(\Delta(P_{<b}), \mathbb{k}) \cong \mathbb{k}$ as P is an hcw-poset, a contradiction. Thus, I is rigid.

Suppose now that I is a rigid monomial ideal. For $a \in P$, we will show that $P_{<a}$ is a homology sphere by induction on $d = d(a)$. If $d = 0$ then $\Delta(P_{<a})$ is a non-empty simplicial complex when a is not the degree of a minimal generator of I .

Thus, $\tilde{H}_{-1}(\Delta(P_{<a}), \mathbb{k}) = 0$. On the other hand, when a is the degree of a minimal generator of I we have $\tilde{H}_{-1}(\Delta(P_{<a}), \mathbb{k}) \cong \mathbb{k}$, as desired. Suppose $d \geq 1$ and that for all $b \in P$ with $d(b) < d$, we have the desired isomorphisms $\tilde{H}_{d(b)-1}(\Delta(P_{<b}), \mathbb{k}) \cong \mathbb{k}$ and $\tilde{H}_i(\Delta(P_{<b}), \mathbb{k}) = 0$ for $i \neq d(b) - 1$. If the element $a \in P$ has $d(a) = d$, then since I is rigid, there exists i such that $\tilde{H}_j(\Delta(P_{<a}), \mathbb{k})$ is isomorphic to \mathbb{k} when $j = i$, and vanishes for $j \neq i$. We must show that $i = d - 1$. We certainly have $i \leq d - 1$ since $P_{<a}$ is $(d - 1)$ -dimensional. Aiming for a contradiction, suppose $i < d - 1$. The structure of the order complex of $P_{<a}$ guarantees that there exists $b < a \in P$ with $d(b) = i + 1$. Using the fact that $b \in P$, we have $\tilde{H}_i(\Delta(P_{<b}), \mathbb{k}) \cong \mathbb{k}$. However, $b < a$ and both having nonzero homology in the same dimension i contradicts rigidity condition (R2). Thus $i < d - 1$ is impossible, making P an hcw-poset. \square

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